Bacalaureat

Matematica M_mate-info

Subiecte rezolvate – Simulare Bacalaureat 2013 – Bucuresti 1.02.2013,

M_mate-info

Subiectul I

1. Rezolvati in multimea numerelor reale ecuatia $|x - 2| = 3x + 6$.

Rezolvare:

Daca $x < 2 \Rightarrow |x - 2| = 2 - x \Rightarrow |x - 2| = 3x + 6 \Leftrightarrow 2 - x = 3x + 6 \Leftrightarrow 4x = -4 \Leftrightarrow x = -1 \in (-\infty, 2)$.

Daca $x \geq 2 \Rightarrow |x - 2| = x - 2 \Rightarrow |x - 2| = 3x + 6 \Leftrightarrow x - 2 = 3x + 6 \Leftrightarrow 2x = -8 \Leftrightarrow x = -4 \notin [2, +\infty)$.

Deci S = {-1}.

2. Rezolvati in multimea numerelor reale inecuatia $\frac{2}{x+4} \leq \frac{1}{3}$.

Rezolvare:

Solutie:

$$\frac{2}{x+4} \leq \frac{1}{3} \Leftrightarrow \frac{2}{x+4} - \frac{1}{3} \leq 0 \Leftrightarrow \frac{6 - x - 4}{3(x+4)} \leq 0 \Leftrightarrow \frac{2 - x}{x+4} \leq 0.$$

x	$-\infty$					-4				2					$+\infty$
$2 - x$	+	+	+	+	+		+	+		0	-	-	-	-	-
$x + 4$	-	-	-	-		0	+	+	+		+	+	+	+	+
$\frac{2-x}{x+4}$	-	-	-	-		|	+	+	+	0	-	-	-	-	-

$x \in (-\infty, -4) \cup [2, +\infty)$.

3. Determinati numarul complex z care are proprietatea $2z + \bar{z} = 3 + 2i$.

Rezolvare:

Notam $z = a + bi$, $a, b \in R \Rightarrow \bar{z} = a - bi \Rightarrow 2z + \bar{z} = 3 + 2i \Leftrightarrow 2(a + bi) + a - bi = 3 + 2i \Leftrightarrow$

$\Leftrightarrow 3a + bi = 3 + 2i \Rightarrow \begin{cases} 3a = 3 \\ b = 2 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = 2 \end{cases}$

Deci z = 1 + 2i.

4. Determinati numarul submultimilor multimii {1, 2, 3, 4, 5, 6} care nu contin elementele 1 si 2.

Rezolvare:

Determinarea numarului submultimilor multimii {1, 2, 3, 4, 5, 6} care nu contin elementele 1si 2 se reduce la a determina numarul submultimilor multimii {3, 4, 5, 6}.

Numarul acestor submultimi este $2^4 = 16$.
Deci sunt 16 submultimi ale multimii {1, 2, 3, 4, 5, 6} care nu contin elementele 1si 2.

<u>Observatie</u>: Daca nu stiti ca numarul submultimilor unei multimi A cu n elemente este 2^n puteti folosi urmatoarea metoda pentru a-l determina:

Submultimile multimii A sunt:

- **multimea vida Ø – o submultime ,**
- **submultimile formate dintr-un element – n submultimi,**
- **submultimi formate din 2 elemente - C_n^2 submultimi,**
- **..,**
- **submultimi formate din n-1 elemente - C_n^{n-1} submultimi,**
- **multimea A**

Deci numarul submultimilor lui A este $1 + n + C_n^2 + ... + C_n^{n-1} + 1 =$
$= C_n^0 + C_n^1 + C_n^2 + ... + C_n^{n-1} + C_n^n = 2^n$

5. Rezolvati ecuatia sin x – cos 2x = 0, daca $x \in \left(0, \frac{\pi}{2}\right)$.

Rezolvare:

Folosim: $\cos 2x = 1 - 2\sin^2 x$

$\sin x - \cos 2x = 0 \Leftrightarrow \sin x - 1 + 2\sin^2 x = 0$. Notam sin x = y. $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow y \in (0, 1)$

Obtinem: $2y^2 + y - 1 = 0 \Rightarrow \Delta = 1 + 8 = 9 \Rightarrow y_{1,2} = \frac{-1 \pm 3}{4} \Rightarrow y_1 = -1 \notin (0, 1), y_2 = \frac{1}{2} \in (0, 1)$

$\sin x = \frac{1}{2}$ si $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow x = \frac{\pi}{6}$

6. Fie A(2, 3), B(- 2, 4), C(4, 6) . Sa se determine ecuatia dreptei d_1 ce trece prin punctul A si este paralela cu dreapta BC.

Rezolvare:

Fie m_{BC} si m_{d_1} pantele dreptelor BC, respectiv d_1 .

$$BC \parallel d_1 \Rightarrow m_{d_1} = m_{BC}.\ m_{BC} = \frac{y_C - y_B}{x_C - x_B} \Leftrightarrow m_{BC} = \frac{6-4}{4-(-2)} \Leftrightarrow m_{BC} = \frac{2}{6} \Leftrightarrow m_{BC} = \frac{1}{3} \Rightarrow m_{d_1} = \frac{1}{3}$$

Ecuatia dreptei d este: $y - y_A = m_{d_1}(x - x_A) \Leftrightarrow y - 3 = \frac{1}{3}(x-2) \Leftrightarrow 3y - 9 = x - 2 \Leftrightarrow x - 3y + 7 = 0.$

Subiectul II

1. Fie $X = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in \mathcal{M}_2(R)$, **a, b** $\neq 0$.

a) **Aratati ca** $X^2 - 2aX + a^2 I_2 = O_2$. **(5p)**

b) **Calculati** X^n. **(5p)**

c) **Sa se arate ca X este inversabila si sa se calculeze** X^{-1}. **(5p)**

Rezolvare:

. a) $X^2 - 2aX + a^2 I_2 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} - 2a\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + a^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$

$= \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix} - \begin{pmatrix} 2a^2 & 2ab \\ 0 & 2a^2 \end{pmatrix} + \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix} = \begin{pmatrix} a^2 - 2a^2 + a^2 & 2ab - 2ab \\ 0 & a^2 - 2a^2 + a^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_2$

b) $X^2 = \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix}$,

$X^3 = X^2 \cdot X = \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{pmatrix}$. (1)

Aratam prin inductie ca $X^n = \begin{pmatrix} a^n & na^{(n-1)}b \\ 0 & a^n \end{pmatrix}$ **pentru orice numar natural n > 1.**

Presupunem ca $X^n = \begin{pmatrix} a^n & na^{(n-1)}b \\ 0 & a^n \end{pmatrix}$. **Aratam ca** $X^{n+1} = \begin{pmatrix} a^{n+1} & (n+1)a^n b \\ 0 & a^{n+1} \end{pmatrix}$.

$X^{n+1} = X^n \cdot X = \begin{pmatrix} a^n & na^{(n-1)}b \\ 0 & a^n \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^{n+1} & (n+1)a^n b \\ 0 & a^{n+1} \end{pmatrix}$. (2)

Din (1) si (2) rezulta ca $X^n = \begin{pmatrix} a^n & na^{(n-1)}b \\ 0 & a^n \end{pmatrix}$ $\forall\ n \in N^*$.

c) det X = $\begin{vmatrix} a & b \\ 0 & a \end{vmatrix}$ = $a^2 \neq 0$ pentru $a \neq 0$. Deci matricea este inversabila.

${}^tX = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$. Calculam complementii algebrici $X_{ij} = (-1)^{i+j}\Delta_{ij}$ ai elementelor transpusei:

$X_{11} = (-1)^{1+1} a_{22} = a$, $X_{12} = (-1)^{1+2} a_{21} = -b$, $X_{21} = (-1)^{2+1} a_{12} = 0$, $X_{22} = (-1)^{2+2} a_{11} = a$.

Calculam $X^* = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} \Rightarrow X^{-1} = \frac{1}{\det(X)} X^* = \frac{1}{a^2}\begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & \frac{-b}{a^2} \\ 0 & \frac{1}{a} \end{pmatrix}$.

2. Pe multimea R se defineste legea de compozitie x*y = x + y + xy .
a) Verificati daca legea de compozitie „*” este asociativa. (5p)
b) Aratati ca legea de compozitie „*” admite element neutru. (5p)
c) Sa se calculeze $1 * \frac{1}{2} * \frac{1}{3} * \ldots * \frac{1}{n-1} * \frac{1}{n}$, $n \in N$. (5p)

Rezolvare:
2. a) asociativitatea:
Fie x, y, z$\in$R oarecare
$(x*y)*z = x*y + z + (x*y)z = x + y + xy + z + (x + y + xy)z = x + y + z + xy + xz + yz + xyz$
$x*(y*z) = x + y*z + x(y*z) = x + (y + z + yz) + x(y + z + yz) = x + y + z + xy + xz + yz + xyz$

Deci $(x*y)*z = x*(y*z)$ $\forall$ x, y, z$\in$R $\Rightarrow$ legea ”*” este asociativa.

b) Fie e$\in$R si x$\in$R oarecare. $e*x = x \Leftrightarrow e + x + ex = x \Leftrightarrow e(x + 1) = 0\ \forall x \in R \Rightarrow e = 0$
Deci $\forall x \in R$ avem $0*x = x$. Verificam ca $x*0 = x\ \forall x \in R$.
$x*0 = x \Leftrightarrow x + 0 + x\cdot 0 = x \Leftrightarrow 0 = 0$ adevarata. Deci $0*x = x*0 = x\ \forall x \in R$.
Deci e = 0 este elementul neutru al legii „*”.

c) $1 * \frac{1}{2} * \frac{1}{3} * \ldots * \frac{1}{n-1} * \frac{1}{n}$, $n \in N$

$1 * \frac{1}{2} = 1 + \frac{1}{2} + 1\cdot\frac{1}{2} = 2$

$1 * \frac{1}{2} * \frac{1}{3} = \left(1 * \frac{1}{2}\right) * \frac{1}{3} = 2 * \frac{1}{3} = 2 + \frac{1}{3} + 2\cdot\frac{1}{3} = 3$

Aratam prin inductie matematica ca $1 * \frac{1}{2} * \frac{1}{3} * \ldots * \frac{1}{n-1} * \frac{1}{n} = n$

Presupunem ca $1 * \frac{1}{2} * \frac{1}{3} * \ldots * \frac{1}{k-1} = k - 1$ si aratam ca $1 * \frac{1}{2} * \frac{1}{3} * \ldots * \frac{1}{k-1} * \frac{1}{k} = k$

$1 * \frac{1}{2} * \frac{1}{3} * \ldots * \frac{1}{k-1} * \frac{1}{k} = (k-1) * \frac{1}{k} = k - 1 + \frac{1}{k} + (k-1)\cdot\frac{1}{k} = k.$

Deci $\forall n \in N$ avem $1 * \frac{1}{2} * \frac{1}{3} * ... * \frac{1}{n-1} * \frac{1}{n} = n$

<u>Subiectul III</u>

1. Se considera functia f: $(0, +\infty) \to R$, $f(x) = \frac{x^3}{3} - \ln x$.

a) Calculati $\lim_{x\to 0} f(x)$. (5p)

a) Sa se determine punctele de extrem ale functiei f. (5p)

b) Sa se demonstreze ca $\ln\sqrt{x} \le \frac{x^2-1}{4}$ pentru orice $x \in (0, +\infty)$. (5p)

Rezolvare:

1. a) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{x^3}{3} - \ln x\right) = \lim_{x\to 0}\frac{x^3}{3} - \lim_{x\to 0}\ln x = 0 - (-\infty) = \infty$

b) Studiem semnul lui f'(x) pe $(0, +\infty)$.

$f'(x) = \frac{3x^2}{3} - \frac{1}{x} = x^2 - \frac{1}{x} = \frac{x^3-1}{x}$. $f'(x) = 0 \Leftrightarrow \frac{x^3-1}{x} = 0 \Rightarrow x^3 - 1 = 0 \Leftrightarrow (x-1)(x^2+x+1) = 0$

$\Rightarrow x_1 = 1$ sau $x^2 + x + 1 = 0$ $\Delta = 1 - 4 = -3 < 0$, $f(1) = \frac{1}{3} - \ln 1 = \frac{1}{3}$.

x	0				1							$+\infty$
f'(x)	-	-	-	-	0	+	+	+	+	+	+	+
f(x)	↘				$\frac{1}{3}$	↗						

Deci f este descrescatoare pe intervalul (0, 1) si f este crescatoare pe intervalul $(1, +\infty)$.
Deci x = 1 este punct de extrem local (minim) pentru f.

c) Consideram functia g: $(0, +\infty) \to R$, $g(x) = \frac{x^2-1}{4} - \ln\sqrt{x}$. Studiem semnul lui g'(x).

$g'(x) = \frac{2x}{4} - \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{x}{2} - \frac{1}{2x} = \frac{x^2-1}{2x}$, $g'(x) = 0 \Leftrightarrow \frac{x^2-1}{2x} = 0 \Rightarrow x^2 - 1 = 0 \Leftrightarrow (x-1)(x+1) = 0$

$\Rightarrow x_1 = 1, x_2 = -1$. Cum $x \in (0, +\infty)$ singura solutie este $x_1 = 1$, $g(1) = \frac{1-1}{4} - \ln 1 = 0$.

x	0				1							$+\infty$
g'(x)	-	-	-	-	0	+	+	+	+	+	+	+
g(x)	↘				0	↗						

Deci g este descrescatoare pe intervalul (0, 1) si g este crescatoare pe intervalul $(1, +\infty)$.
Deci x = 1 este punct de extrem local pentru g $\Rightarrow$ $g(x) \ge g(1)$ $\forall x \in (0, +\infty) \Rightarrow$

$\Rightarrow \frac{x^2-1}{4} - \ln\sqrt{x} \geq 0 \ \forall x \in (0, +\infty) \Rightarrow \ln\sqrt{x} \leq \frac{x^2-1}{4}$ **pentru orice** $x \in (0, +\infty)$.

2. Fie $f_n : [0, 1] \to R$, $f_n(x) = x^n e^{3x}$ **si** $I_n = \int_0^1 f_n(x)dx$, $n \geq 0$.

a) **Sa se calculeze** I_0 **si** I_1. **(5p)**

b) **Sa se calculeze** $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx$. **(5p)**

c) **Sa se arate ca** $3I_{n+1} + (n+1)I_n = e^3 \ \forall n \geq 0$.

Rezolvare:

2. a) $I_0 = \int_0^1 x^0 e^{3x}dx = \int_0^1 e^{3x}dx = \left.\frac{e^{3x}}{3}\right|_0^1 = \frac{e^3}{3} - \frac{1}{3}$,

$I_1 = \int_0^1 xe^{3x}dx = \int_0^1 x\left(\frac{e^{3x}}{3}\right)'dx = \left. x\frac{e^{3x}}{3}\right|_0^1 - \int_0^1 \frac{e^{3x}}{3}dx = \frac{e^3}{3} - 0 - \frac{1}{3}\left.\frac{e^{3x}}{3}\right|_0^1 = \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} = \frac{2e^3}{9} + \frac{1}{9}$.

b) $f_n : [0, 1] \to R$, $f_n(x) = x^n e^{3x}$ **este o functie continua. Aplicand teorema de medie** $\exists c$, $0 \leq c \leq 1$ **astfel incat** $\int_0^1 f_n(x)dx = f_n(c)(1-0) = c^n e^{3c} \Rightarrow \lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = \lim\limits_{n\to\infty} c^n e^{3c} = 0$.

c) $I_{n+1} = \int_0^1 x^{n+1}e^{3x}dx = \int_0^1 x^{n+1}\left(\frac{e^{3x}}{3}\right)'dx = \left. x^{n+1}\frac{e^{3x}}{3}\right|_0^1 - \int_0^1 (n+1)x^n \frac{e^{3x}}{3}dx = \frac{e^3}{3} - 0 - \frac{n+1}{3} I_n \Rightarrow$

$\Rightarrow I_{n+1} = \frac{e^3}{3} - \frac{n+1}{3} I_n \Leftrightarrow 3I_{n+1} = e^3 - (n+1)I_n \Leftrightarrow 3I_{n+1} + (n+1)I_n = e^3 \ \forall n \geq 0$.